

A new linear quotient of \mathbb{C}^4 admitting a symplectic resolution

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Abstract We show that the quotient \mathbb{C}^4/G admits a symplectic resolution for $G = Q_8 \times_{\mathbb{Z}/2} D_8 < \mathrm{Sp}_4(\mathbb{C})$. Here Q_8 is the quaternionic group of order eight and D_8 is the dihedral group of order eight, and G is the quotient of their direct product which identifies the nontrivial central elements – Id of each. It is equipped with the tensor product representation $\mathbb{C}^2 \boxtimes \mathbb{C}^2 \cong \mathbb{C}^4$. This group is also naturally a subgroup of the wreath product group $Q_8^2 \rtimes S_2 < \mathrm{Sp}_4(\mathbb{C})$. We compute the singular locus of the family of commutative spherical symplectic reflection algebras deforming \mathbb{C}^4/G . We also discuss preliminary investigations on the more general question of classifying linear quotients V/G admitting symplectic resolutions.

Keywords Symplectic resolution · Symplectic smoothing · Symplectic reflection algebra · Poisson algebra · Poisson variety · Symplectic leaves · Quotient singularity · McKay correspondence

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1 Introduction and main results

The quotients V/G , for $G < \mathrm{Sp}(V)$ a finite subgroup, which admit a symplectic resolution (this notion is recalled in the next subsection) are known to include:

- (i) The type A_{n-1} Weyl groups S_n , acting on $V = T^*\mathfrak{h}$, where \mathfrak{h} is the reflection representation; here a resolution is given by the reduced Hilbert scheme $\mathrm{Hilb}_0^{n-1} \mathbb{C}^2$;
- (ii) The wreath product groups $H^n \rtimes S_n$, for $H < \mathrm{SL}_2(\mathbb{C})$ a finite subgroup, acting on \mathbb{C}^{2n} ; here a resolution is given by the Hilbert scheme $\mathrm{Hilb}^n \widetilde{\mathbb{C}^2/H}$, where $\widetilde{\mathbb{C}^2/H} \rightarrow \mathbb{C}^2/H$ is the minimal resolution of the Kleinian (or du Val) singularity \mathbb{C}^2/H ;
- (iii) The exceptional complex reflection group $G_4 < \mathrm{GL}_2(\mathbb{C}) < \mathrm{Sp}_4(\mathbb{C})$, see [3].

The main purpose of this paper is to add one more example to this list:

- (iv) The group $G = Q_8 \times_{\mathbb{Z}/2} D_8$, where $Q_8 < \mathrm{SL}_2(\mathbb{C})$ is the quaternionic group of order eight, $D_8 < \mathrm{O}_2(\mathbb{C})$ is the dihedral group of order eight, and $Q_8 \times_{\mathbb{Z}/2} D_8$ is the quotient of their product which identifies the centers of Q_8 and D_8 , acting on the tensor product representation $\mathbb{C}^2 \boxtimes \mathbb{C}^2$.

As we will discuss briefly in Sect. 1.4 below, we suspect there are few (if any) other examples remaining to be discovered.

Remark 1.0.1 In cases (i) and (ii) above, one can construct the symplectic resolution in a natural way by a certain Hamiltonian reduction procedure. On the other hand, in case (iii), we do not know of such a construction (although Lehn and Sorger constructed in [19] a resolution in a more explicit computational manner). We have also been unable to find such a construction for our new example (iv). To find such a construction seems like an interesting problem.

In what follows, we will provide more detailed explanations of the above and explain the proof that (iv) admits a symplectic resolution, up to a computation given in Sect. 3.

1.1 Symplectic resolutions

A symplectic resolution $\pi : \tilde{X} \rightarrow X$ of a (singular) variety X is a (smooth) symplectic variety \tilde{X} equipped with a proper, birational map π to X . We are particularly interested in the case that X is affine; in this case π can also be viewed as an “affinization” of the symplectic variety \tilde{X} . Such structures have attracted a lot of interest in the last decade: see, e.g., [11, 17], and have strong applications to representation theory, quantum algebra, algebraic geometry and symplectic geometry. Examples include:

- the Springer resolution $\rho : T^*(G/B) \rightarrow \mathcal{N}$ of the nilpotent cone \mathcal{N} , as well as its restriction to resolutions $\rho^{-1}(S \cap \mathcal{N}) \rightarrow (S \cap \mathcal{N})$, where S is a Kostant–Slodowy slice at a nilpotent element $e \in \mathcal{N}$ to the coadjoint orbit $\mathrm{Ad} G(e)$;
- the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points on a symplectic surface S , resolving its n th symmetric power $\mathrm{Sym}^n(S)$;
- Nakajima quiver varieties;
- when $S = \widetilde{\mathbb{C}^2/G}$ is a minimal resolution of a Kleinian (or du Val) singularity \mathbb{C}^2/G , then $\mathrm{Hilb}^n(S)$ resolves the affine singularity $\mathrm{Sym}^n(\mathbb{C}^2/G)$ (this is example (ii) of the previous subsection); and
- hypertoric varieties.

The symplectic structure on \tilde{X} naturally endows X with a Poisson structure. Conversely, if X is a Poisson variety, we say that it admits a symplectic resolution if there exists a resolution \tilde{X} as above, such that π is a Poisson morphism. It is an interesting question to determine which Poisson varieties admit symplectic resolutions—this is a very strong condition. On the other hand, when such resolutions exist, they are derived unique: by [16], any two symplectic resolutions of a Poisson variety have equivalent derived categories of coherent sheaves.

In the case $X = \mathbf{C}^{2n}/G$, $G < \mathbf{Sp}_{2n}(\mathbf{C})$, the only known examples where X admits a symplectic resolution are the cases (i)–(iii) of the previous subsection, and products thereof.

We exhibit a new example of a linear symplectic quotient admitting a symplectic resolution: \mathbf{C}^4/G , where $G = Q_8 \times_{\mathbf{Z}/2} D_8$, where $Q_8 < \mathbf{SL}_2(\mathbf{C})$ is the quaternionic group of order eight, $D_8 < \mathbf{O}_2(\mathbf{R}) < \mathbf{O}_2(\mathbf{C})$ is the dihedral group of order eight, and G is the quotient of their direct product identifying the nontrivial central elements $-\text{Id}$ of each. This group G is equipped with the faithful tensor product representation $\mathbf{C}^4 = \mathbf{C}^2 \boxtimes \mathbf{C}^2$. Since Q_8 preserves a symplectic form on \mathbf{C}^2 and D_8 preserves an orthogonal form on \mathbf{C}^2 , their product naturally preserves a symplectic form on the tensor product \mathbf{C}^4 . Thus G is naturally a subgroup of $\mathbf{Sp}_4(\mathbf{C})$. This group can also be realized explicitly as the following subgroup of the wreath product $Q_8^2 \rtimes S_2$:

$$G = \{(\pm g) \oplus g, ((\pm g) \oplus g)\sigma \mid g \in Q_8\} < Q_8^2 \rtimes S_2, \quad (1.1.1)$$

where $\sigma \in S_2$ is the nontrivial permutation.

Our main result is then

Theorem 1.1.2 *The quotient \mathbf{C}^4/G admits a symplectic resolution $\widetilde{\mathbf{C}^4/G} \rightarrow \mathbf{C}^4/G$.*

1.2 Symplectic reflection algebras

The proof of Theorem 1.1.2 is based on techniques from [7] on the representation theory of symplectic reflection algebras, together with a theorem of Namikawa [21], and is similar to that used in [13, §7] and [3, §4]. Namely, by Namikawa's result, since \mathbf{C}^4/G has a contracting \mathbf{C}^\times -action and the Poisson bracket has negative degree with respect to this action, the existence of a symplectic resolution follows from the existence of a smooth filtered Poisson deformation of $\mathbf{C}[V^*]^G$. Natural candidates for such a deformation are the commutative spherical symplectic reflection algebras $eH_c(G)e$ of [7], where $H_c(G)$ is the symplectic reflection algebra of *op. cit.* (with $t = 0$, where t is as in *op. cit.* or Sect. 3.1 below), which deforms the skew product algebra $\mathbf{C}[V^*] \rtimes \mathbf{C}[G]$, and $e \in \mathbf{C}[G]$ is the symmetrizer idempotent $e = \frac{1}{|G|} \sum_{g \in G} g$. Conversely, by [12, Corollary 1.21], the existence of a symplectic resolution of V/G implies that the algebras $eH_c(G)e$ are generically smooth. We deduce

Theorem 1.2.1 [12, 21] *The following conditions are equivalent:*

- (i) V/G admits a symplectic resolution;
- (ii) There exists a smooth commutative spherical symplectic reflection algebra $eH_c(G)e$;
- (iii) The algebras $eH_c(G)e$ are smooth for generic c .

We remark that the equivalence of (ii) and (iii) is also clear since smoothness is an open condition in c . We will prove

Theorem 1.2.2 *For $G = Q_8 \times_{\mathbf{Z}/2} D_8$, $eH_c(G)e$ is smooth for generic parameters c .*

Later, in Sect. 4, we will prove a much more general result, which completely classifies the parameters c for which the algebra $eH_c(G)e$ is smooth (Theorem 4.2.1), which turns out to

be the complement of exactly 21 hyperplanes. There, we will also describe in more detail the singular locus of the varieties $\mathrm{Spec} eH_c(G)e$.

Recall that, for general $G < \mathrm{Sp}(V)$, commutative spherical symplectic reflection algebras $eH_c(G)e$ are parameterized by class functions $c : \mathbf{C}[G] \rightarrow \mathbf{C}$ (i.e., conjugation-invariant functions) which are supported on the symplectic reflections $S \subseteq G$, i.e., those elements $s \in G$ such that $s - \mathrm{Id}$ has rank two. In our example, there are five conjugacy classes of such elements, so the parameter space is five-dimensional.

To prove Theorem 1.2.2, we use the following reformulations of smoothness for commutative spherical symplectic reflection algebras, at least some of which are well known:

Proposition 1.2.3 *The following conditions are equivalent for a commutative spherical symplectic reflection algebra $eH_c(G)e$:*

- (i) $eH_c(G)e$ is smooth;
- (ii) $H_c(G)$ admits no irreducible representations which, as G -representations, are proper subrepresentations of the regular representation;
- (iii) $H_c(G)$ admits no irreducible representations of dimension strictly less than $|G|$;
- (iv) All finite-dimensional representations of $H_c(G)$ are, as G -representations, direct sums of finitely many copies of the regular representation;
- (v) All irreducible representations of $H_c(G)$ restrict to the regular representation of G .

Proof Clearly (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii). By [7, Theorem 3.1], there is a Satake isomorphism $Z(H_c(G)) \xrightarrow{\sim} eH_c(G)e$ given by $z \mapsto z \cdot e$, where $Z(H_c(G))$ is the center of $H_c(G)$. Therefore (i) is equivalent to $Z(H_c(G))$ being smooth.

By [7, Theorem 1.7], for every character $\eta : Z(H_c(G)) \rightarrow \mathbf{C}$ contained in the smooth locus of $\mathrm{Spec} Z(H_c(G))$, the quotient $H_c(G)/\ker(\eta)$ is a matrix algebra with unique simple representation $H_c(G)e/\ker(\eta)e \cong \mathbf{C}[G]$. Therefore (i) implies (v).

Since the smooth locus is dense in $\mathrm{Spec} Z(H_c(G))$, [7, Theorem 1.7] also implies that the P.I. degree of $H_c(G)$ equals $|G|$. Hence (iii) implies that the Azumaya locus of $H_c(G)$ equals $\mathrm{Spec} Z(H_c(G))$. However, it is known, e.g. [14, Theorem 4.8], that the smooth locus of $\mathrm{Spec} Z(H_c(G))$ equals the Azumaya locus of $H_c(G)$ over $Z(H_c(G))$. Therefore (iii) implies (i).

Using Lemma 4.4.1 below, we can show also that (ii) implies (i).¹ Suppose that (ii) holds. By Lemma 4.4.1, for every point of $\mathrm{Spec} Z(H_c(G))$, i.e., for every character η of $Z(H_c(G))$, there exists a representation M of $H_c(G)$ isomorphic to the regular representation with central character η . By (ii), this must be irreducible. Because the P.I. degree of $H_c(G)$ equals $|G|$, again η must be in the Azumaya locus and hence a smooth point. Thus (ii) implies (i). \square

We will prove Theorem 1.2.2 by demonstrating that condition (iii) holds for certain values of c (and hence also for generic c). We will not need (ii) for Theorem 1.2.2, but will use it in the proof of the stronger Theorem 4.2.1.

1.3 Restrictions on the G -character of representations of symplectic reflection algebras

To show that condition (iii) holds for generic c (or equivalently, some value of c), we exhibit sufficiently many restrictions on the G -character χ of finite-dimensional representations of $H_c(G)$. These restrictions apply to arbitrary symplectic reflection algebras.

¹ Since we will only actually need this implication for Theorem 4.2.1 and not for Theorem 1.2.2, we postponed Lemma 4.4.1 used here to Sect. 4.4.

For now, let $G < \mathbf{Sp}(V)$ be an arbitrary finite subgroup, for an arbitrary symplectic vector space V . Let $H_c(G)$ be a symplectic reflection algebra deforming $\mathbf{C}[V^*] \rtimes G$, and let $\rho : H_c(G) \rightarrow \text{End}(U)$ be a finite-dimensional representation. Whenever $x, y \in V$, then $[x, y] \in \mathbf{C}[G]$, and evidently $\text{tr}(\rho([x, y])) = 0$. This means that the character $\chi := \text{tr} \circ \rho|_{\mathbf{C}[G]}$ of U annihilates all commutators $[x, y]$. These commutators are certain explicit elements of $\mathbf{C}[G]$ supported on \mathcal{S} , that we will describe later.

To show that χ must be a multiple of the regular character, i.e., that $\chi(g) = 0$ for all nontrivial g , such restrictions cannot be sufficient unless all nontrivial elements of G are symplectic reflections. This only happens when $G < \mathbf{SL}_2(\mathbf{C})$.² To obtain more restrictions, we observe that, whenever $g \in G$, x is a fixed vector of g , and $y \in V$ is another element, then $[x, gy] = g[x, y] \in \mathbf{C}[G]$, so that $\chi(g[x, y]) = 0$ as well. We have deduced:

Proposition 1.3.1 *Let $H_c(G)$ be a symplectic reflection algebra associated to $G < \mathbf{Sp}(V)$. Let $g \in G$, $x \in V^g$, and $y \in V$. Then the element $g[x, y] \in H_c(G)$ lies in $\mathbf{C}[G]$, and is annihilated by all characters of finite-dimensional representations of $H_c(G)$.*

Then, the proof of Theorem 1.2.2, and hence also Theorem 1.1.2, is completed by a computation of the elements $g[x, y]$ that can arise in the case $G = Q_8 \times_{\mathbf{Z}/2} D_8$; together with the above proposition this will imply that condition (iii) of Proposition 1.2.3 holds. We do this in Sect. 3 below.

Remark 1.3.2 The above proposition provides an algorithm for restricting the characters of finite-dimensional representations of $H_c(G)$ for generic c . In fact, this was how we discovered our theorem in the first place. However, note that for the example of $G = G_4 < \mathbf{GL}_2(\mathbf{C}) < \mathbf{Sp}_4(\mathbf{C})$, as computed in [3, §4], the algorithm above only restricts the G -representations to be a direct sum of copies of two representations (denoted E and F in *op. cit.*), of dimension less than $|G|$. Therefore these restrictions are not, in general, exhaustive, and do not give a necessary condition for V/G to admit a symplectic resolution (since \mathbf{C}^4/G_4 does admit a resolution by [3]).

1.4 On the (non) existence of symplectic resolutions for other linear symplectic quotients

In this section, we explain what we know about the question of which finite groups $G < \mathbf{Sp}(V)$ have the property that V/G admits a symplectic resolution, which we would like to address in future work.

By [23], it is known that a linear symplectic quotient V/G by a finite subgroup $G < \mathbf{Sp}(V)$ can only admit a symplectic resolution if G is generated by symplectic reflections. In the case that G preserves a Lagrangian subspace U , so $G < \mathbf{GL}(U) < \mathbf{Sp}(V)$, i.e., G is a complex reflection group, these have a well known classification by Shephard and Todd [22]. It was shown, first for finite Coxeter groups in [13], and then for all complex reflection groups in [3] that, aside from one exceptional group, denoted by G_4 , only the infinite families already mentioned (Weyl groups S_{n+1} and wreath products $(\mathbf{Z}/m)^n \rtimes S_n$) have the property that V/G admits a symplectic resolution.

On the other hand, there are many groups generated by symplectic reflections that are not complex reflection groups. These groups have been classified in [6]. Aside from finitely many exceptional groups, they fall into infinite families. These infinite families are subgroups of wreath products $\Gamma^n \rtimes S_n$, where Γ is a finite subgroup of $\mathbf{SL}_2(\mathbf{C})$ of type D or E : when

² On the other hand, in this case, one can indeed deduce that condition (iii) of Proposition 1.2.3 holds for generic c , which are just class functions supported away from the trivial element of G , and this gives another proof of the well known fact that \mathbf{C}^2/G admits a symplectic resolution.

$\dim V = 4$ there are several families of symplectic subgroups of $\Gamma^2 \rtimes S_2$, and there are also a few families of subgroups that exist for even dimensions ≥ 4 . Our group $G = Q_8 \times_{\mathbb{Z}/2} D_8$ is included in the latter list.

Preliminary (but not definitive) computer evidence we have considered seems to suggest that, for the infinite families involving $\dim V > 4$, and many of the infinite families in the case $\dim V = 4$, there is no smooth commutative spherical symplectic reflection algebra deforming V/G and hence no symplectic resolution. The problem essentially reduces to the case of the families in $\dim V = 4$, because the infinite families all contain parabolic subgroups $K < G$ such that $\dim(V^K)^\perp = 4$, and then one can adapt Losev's work [18] to show that, if V/G admits a smooth deformation by a commutative spherical symplectic reflection algebra, so must $(V^K)^\perp/K$ as well. In these cases, K is in one of the infinite families for the case of dimension four, so (except when K is our group $Q_8 \times_{\mathbb{Z}/2} D_8$), one reduces to showing that \mathbb{C}^4/K admits no smooth deformation by a commutative spherical symplectic reflection algebra.

We would guess that our group $G = Q_8 \times_{\mathbb{Z}/2} D_8$ is the only group in any of Cohen's aforementioned infinite families [aside from the wreath products of groups in $\mathrm{SL}_2(\mathbb{C})$] such that V/G admits a symplectic resolution. We do not presently have any understanding of the (finitely many) exceptional symplectic reflection groups on Cohen's list that are not complex reflection groups.

2 The group $Q_8 \times_{\mathbb{Z}/2} D_8$

It is useful to describe the group $G = Q_8 \times_{\mathbb{Z}/2} D_8$ in some more detail—it turns out to enjoy some remarkable properties.

Let $i \in \mathbb{C}$ denote the usual “imaginary” number, i.e., $i^2 = -1$. Let

$$Q_8 := \{\pm \mathrm{Id}, \pm I, \pm J, \pm K \mid IJ = K, JK = I, KI = J, I^2 = J^2 = K^2 = -\mathrm{Id}\} < \mathrm{SL}_2(\mathbb{C})$$

be the usual description of Q_8 . A faithful representation is given by:

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Let

$$D_8 := \{\mathrm{Id}, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3 \mid \sigma^2 = \mathrm{Id} = \rho^4, \sigma\rho\sigma = \rho^{-1}\} < \mathrm{O}_2(\mathbb{C})$$

be the usual description of D_8 . A faithful representation is given by:

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the centers of Q_8 and D_8 are both $\{\pm \mathrm{Id}\} = \{\rho^2, \mathrm{Id}\}$ (which also coincide with the subgroup of scalar matrices, since \mathbb{C}^2 is an irreducible representation of both). This makes $Q_8 \times_{\mathbb{Z}/2} D_8$ act on $\mathbb{C}^2 \boxtimes \mathbb{C}^2 \cong \mathbb{C}^4$, preserving the product of the symplectic form on the first factor and the orthogonal form on the second factor (as pointed out in the introduction). That is, it preserves a symplectic form on \mathbb{C}^4 , and this identifies $G := Q_8 \times_{\mathbb{Z}/2} D_8 < \mathrm{Sp}_4(\mathbb{C})$. We will refer to the defining representation \mathbb{C}^4 as the *symplectic reflection representation*. It is clear that it is irreducible.

We now collect the facts we will need about G :

Proposition 2.0.1 (i) *All conjugacy classes of G , except for $\{\text{Id}\}$ and $\{-\text{Id}\}$, are of order two and of the form $\{\pm(g, h)\}$.*

(ii) *The symplectic reflections in $Q_8 \times_{\mathbf{Z}/2} D_8$ are the noncentral elements (g, h) where $g \in Q_8$ and $h \in D_8$ have the same order (two or four).*

(iii) *Equivalently, the symplectic reflections are exactly the noncentral elements of order two.*

(iv) *Explicitly, there are five conjugacy classes of symplectic reflections:*

$$\{\pm(I, \rho)\}, \{\pm(J, \rho)\}, \{\pm(K, \rho)\}, \{\pm(\text{Id}, \sigma)\}, \{\pm(\text{Id}, \sigma\rho)\}.$$

(v) *The group G has seventeen irreducible representations over \mathbf{C} ; sixteen of them are one dimensional and the other is the symplectic reflection representation \mathbf{C}^4 .*

Proof (i) It is clear that the conjugacy class $\text{Ad } G\{(g, h)\}$ containing an element (g, h) is the product of conjugacy classes of $(g, 1)$ and $(1, h)$, i.e., $\text{Ad}(Q_8)\{g\} \times_{\mathbf{Z}/2} \text{Ad}(D_8)\{h\}$. The statement follows from the fact that it holds for each of Q_8 and D_8 .

(ii, iv) The eigenvalues of (g, h) are the four pairwise products of an eigenvalue of g and an eigenvalue of h . In order for the result to contain one as an eigenvalue, therefore, g and h^{-1} must share a common eigenvalue. In this case, this can only happen if the eigenvalues of g and h are both i and $-i$ (i.e., g and h both have order four), or if $g = \pm \text{Id}$ and $h \in \{\sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$.

(iii) Note that, if one of g and h has order four, but the other has order two, then $(g, h)^2 = -\text{Id}$, so (g, h) has order four as well. So the description follows.

(v) There are sixteen one-dimensional representations since $G/[G, G]$ has order sixteen. Since the sum of squares of the dimensions of the irreducible representations must equal the order, 32, of G , these together with the (four-dimensional) symplectic reflection representation must be all of the irreducible representations. \square

2.1 Outer automorphisms of G

The material of this section will not be needed in the paper, but we are including it to demonstrate the unique symmetry of G (which, along with properties already described, makes it appear somewhat exceptional).

Proposition 2.1.1 (i) *The permutation action of $\text{Out}(G)$ on the conjugacy classes of symplectic reflections defines an isomorphism*

$$\text{Out}(G) \xrightarrow{\sim} S_5. \quad (2.1.2)$$

(ii) *All of the outer automorphisms are obtainable by conjugation by elements of $\text{Sp}_4(\mathbf{C})$.*

(iii) *This outer automorphism group is generated by the outer automorphism group of D_8 along with the conjugation action of $Q_8^2 \rtimes S_2$.*

Proof (i) In the realization $G = Q_8 \times_{\mathbf{Z}/2} D_8$, one sees the subgroup $\text{Out}(Q_8) \times \text{Out}(D_8)$ of order 12 of the outer automorphism group $\text{Out}(G)$. On the other hand, in the realization $G < Q_8^2 \rtimes S_2$, one sees the subgroup of outer automorphisms coming from conjugation by the larger group. Since \mathbf{C}^4 is an irreducible representation of G , the centralizer of G in $Q_8^2 \rtimes S_2$ is the subgroup $\{\pm \text{Id}\}$ of scalar matrices in G . Therefore the quotient of $Q_8^2 \rtimes S_2$ by G embeds in $\text{Out}(G)$. In this way one obtains a subgroup of outer automorphisms of order 4 (it is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$).

We claim that these two groups intersect trivially, and their permutation actions on conjugacy classes of symplectic reflections generate all of S_5 . To see this, note first that, in

the realization $G = Q_8 \rtimes_{\mathbf{Z}/2} D_8$, the outer automorphism subgroup $\text{Out}(Q_8) \times \text{Out}(D_8)$ preserves the partition of symplectic reflection conjugacy classes into the cells

$$\{\{\pm(I, \rho)\}, \{\pm(J, \rho)\}, \{\pm(K, \rho)\}\}, \text{ and } \{\{\pm(\text{Id}, \sigma)\}, \{\pm(\text{Id}, \sigma\rho)\}\}.$$

This produces an isomorphism

$$\text{Out}(Q_8) \times \text{Out}(D_8) \xrightarrow{\sim} S_3 \times S_2 < S_5,$$

by permutations of symplectic reflection conjugacy classes.

On the other hand, in the realization $G < Q_8^2 \rtimes S_2$, the outer automorphism subgroup coming from the conjugation action of $Q_8^2 \rtimes S_2$ preserves the partition of symplectic reflection conjugacy classes into the cells

$$\{\{\pm(I, \rho)\}, \{\pm(J, \rho)\}, \{\pm(K, \rho)\}, \{\pm(\text{Id}, \sigma)\}\}, \text{ and } \{\{\pm(\text{Id}, \sigma\rho)\}\},$$

the last conjugacy class being the one consisting of the noncentral diagonal matrices. This produces an isomorphism

$$\text{Ad}(Q_8^2 \rtimes S_2) / \text{Ad}(G) \xrightarrow{\sim} \mathbf{Z}/2 \times \mathbf{Z}/2 < S_4 < S_5,$$

again by permuting the symplectic reflection conjugacy classes.

Since the image of the above two groups includes the transpositions $(1, 2)$, $(2, 3)$, $(3, 4)$ and $(4, 5)$, they generate the whole of S_5 .

To prove the assertion, it remains to show that one obtains from this an isomorphism $\text{Out}(G) \rightarrow S_5$, by permuting the symplectic reflection conjugacy classes.

First, we have to explain why all outer automorphisms preserve the conjugacy classes of symplectic reflections. This follows from the fact that the symplectic reflections are exactly the noncentral involutions (Proposition 2.0.1.(iii)). Alternatively, since the defining representation \mathbf{C}^4 of G is the unique four-dimensional irreducible representation, any outer automorphism must be obtained by conjugation by an element of $\text{GL}_4(\mathbf{C})$, so that the symplectic reflections (elements g such that $(\mathbf{C}^4)^g$ is two-dimensional) must be preserved.

Hence, the above yields a well defined epimorphism $\text{Out}(G) \twoheadrightarrow S_5$. It remains to show that this is injective, i.e., the kernel of $\text{Aut}(G) \rightarrow S_5$ is the inner automorphism group. It is clear that the inner automorphism group is contained in the kernel, so we only have to show it equals the kernel. Any automorphism which fixes all the symplectic reflection conjugacy classes is determined by how it acts on each of the classes (since G is generated by symplectic reflections). Since there are 5 conjugacy classes, each of order 2, there can be at most 32 of these automorphisms. Again, since G is generated by symplectic reflections, the inner automorphism group acts faithfully on the set of symplectic reflections and hence we get $16 = |G/Z(G)|$ automorphisms this way. Therefore it suffices to show that there are exactly 16 automorphisms of G fixing the conjugacy classes of symplectic reflections. However, any four of these conjugacy classes generates the fifth, which implies that there can be at most 16. Hence there are exactly 16 and the kernel of $\text{Aut}(G) \rightarrow S_5$ is the inner automorphism group, as desired.

(iii) This follows from the proof of (i): we pointed out that all of the mentioned elements generate the whole outer automorphism group S_5 . But more precisely, we did not actually need the outer automorphism group of Q_8 : the outer automorphism group of D_8 provides the transposition in S_5 , and this together with the order-four subgroup of $S_4 < S_5$ (where S_4 does not contain the aforementioned transposition) generates all of S_5 .

(ii) This follows from (iii) if we can just show that the nontrivial element of $\text{Out}(D_8) \cong \mathbf{Z}/2$, as a subgroup of $\text{Out}(G)$, is obtainable by conjugation by an element of $\text{Sp}_4(\mathbf{C})$ (note

that $Q_8^2 \rtimes S_2 < \mathrm{Sp}_4(\mathbf{C})$, which proves that the conjugation action of the latter is by symplectic transformations). This element is the automorphism $\sigma \mapsto \sigma\rho$, $\rho \mapsto \rho$, of order four as an honest automorphism (as an outer automorphism it has order two). It suffices to show that this automorphism of D_8 is given by conjugation by an element of $\mathrm{O}_2(\mathbf{C})$. This can be done by conjugating by any square root of ρ , which is indeed orthogonal. \square

Remark 2.1.3 Alternatively, to show that the automorphism of D_8 is given by conjugation by an element of $\mathrm{O}_2(\mathbf{C})$, one can argue that, since \mathbf{C}^2 is the unique irreducible representation of D_8 of dimension 2, the outer automorphism is given by conjugating by some matrix, and this can be taken to be orthogonal since it can be taken to be real (there is only one real irreducible representation of dimension two).

A similar argument applied to G yields a proof of all of part (ii): the irreducible representation \mathbf{C}^4 is the unique one of dimension four, so any outer automorphism is obtained by conjugation by some element of $\mathrm{GL}_4(\mathbf{C})$. In fact, this is the unique irreducible symplectic representation of dimension four, since all the other irreducible representations of G are one-dimensional and extend to two-dimensional irreducible symplectic representations (the symplectic representation theory of any finite group is completely reducible just like the ordinary representation theory). Thus, any outer automorphism must be given by conjugating by an element of $\mathrm{Sp}_4(\mathbf{C})$.

3 Proof of Theorem 1.2.2

3.1 Recollections on symplectic reflection algebras (following [7])

Let (V, ω) be a symplectic vector space. Recall that a *symplectic reflection* is an element $s \in \mathrm{Sp}(V)$ such that $\mathrm{rk}(s - \mathrm{Id}) = 2$, i.e., $V^s \subseteq V$ is a codimension-two subspace, which we call the reflecting hyperplane of s . The restriction of ω to V^s is nondegenerate, so $V = V^s \oplus (V^s)^\perp$. Let $\pi_{(V^s)^\perp} : V \rightarrow (V^s)^\perp$ be the orthogonal (with respect to ω) projection. Define the (in general degenerate on V) form

$$\omega_s : V \otimes V \rightarrow \mathbf{C}, \quad \omega_s(v, w) = \omega(\pi_{(V^s)^\perp}(v), \pi_{(V^s)^\perp}(w)).$$

Now, let $G < \mathrm{Sp}(V)$ be a finite subgroup. Let $\mathcal{S} \subseteq G$ be the subset of symplectic reflections. Let $\mathbf{C} = \mathbf{C}[\mathcal{S}]^G$ denote the set of conjugation-invariant functions on \mathcal{S} . For every $c \in \mathbf{C}$ and $t \in \mathbf{C}$, define the *symplectic reflection algebra*

$$H_{c,t}(G) := TV \rtimes G \left/ \left(v \cdot w - w \cdot v - t\omega(v, w) - \sum_{s \in \mathcal{S}} c(s)\omega_s(v, w) \right) \right.,$$

where TV is the tensor algebra on V (with multiplication \cdot). As in the introduction, let $e \in \mathbf{C}[G]$ be the symmetrizer element $e := \frac{1}{|G|} \sum_{g \in G} g$, and define the spherical symplectic reflection algebra as $eH_{c,t}(G)e$.

We will be interested in the case $t = 0$, and will use the notation $H_c(G) := H_{c,0}(G)$. In this case, it is a well known result of [7] that $eH_c(G)e$ is commutative and is in fact isomorphic to the center of $H_c(G)$. Therefore, we call $eH_c(G)e$ a commutative spherical symplectic reflection algebra.

3.2 Proof of Theorem 1.2.2

As before, set $G := Q_8 \times_{\mathbb{Z}/2} D_8$. We will prove in the next subsection the following more precise result, using Proposition 1.3.1:

Proposition 3.2.1 *Let χ be the G -character of a finite-dimensional representation of $H_c(G)$. Then the following equations:*

$$\sum_{s \in \mathcal{S}} \chi(s) \cdot c(s) = 0, \quad (3.2.2)$$

and, for all $g \in \mathcal{S}$,

$$2\chi(-\text{Id})c(g) + \sum_{s \in \mathcal{S} \setminus \{g, -g\}} \chi(gs) \cdot c(s) = 0, \quad (3.2.3)$$

are satisfied by χ .

In view of Proposition 1.2.3.(iii), we immediately conclude

Corollary 3.2.4 *If equations (3.2.2) and (3.2.3) are not satisfied for any character χ of a representation of dimension less than $|G|$, then $eH_c(G)e$ is smooth.*

We may conclude from this Theorem 1.2.2:

Proof of Theorem 1.2.2 First note that, by Proposition 2.0.1, for all $h \in G$, either $h \in \mathcal{S}$, or $h = gs$ for some $g, s \in \mathcal{S}$ (and if $h \neq \text{Id}$, then $g \neq s$; recall $s = s^{-1}$ for all $s \in \mathcal{S}$). Therefore, at least one of equations (3.2.2)–(3.2.3) is non-trivial unless $\chi(h) = 0$ for all $h \neq \text{Id}$, i.e., χ is a multiple of the regular character. As a result, (3.2.2) and (3.2.3) define proper linear subspaces of $\mathbb{C}[\mathcal{S}]^G$ as χ ranges over all characters of finite-dimensional representations of dimension less than $|G|$. Hence, for c not in any of these finitely many proper linear spaces (and in particular for generic c), Corollary 3.2.4 implies that $eH_c(G)e$ is smooth. \square

We remark that the Eqs. (3.2.2) and (3.2.3) for c have integer coefficients since all characters of G are integer-valued (and characters of representations of dimension $< |G|$ are valued in integers of absolute value less than $|G|$).

3.3 Proof of Proposition 3.2.1

Again take $G = Q_8 \times_{\mathbb{Z}/2} D_8$ and $V = \mathbb{C}^4$. If χ is the character of a representation of $H_c(G)$, then Proposition 1.3.1 implies that $\chi(g[x, y]) = 0$ whenever $x \in V^g$ and $y \in V$.

Choose $x, y \in V$ such that $\omega(x, y) = 2$. For $s \in \mathcal{S}$, since $s^2 = \text{Id}$, we conclude that $(V^s)^\perp = V^{-s}$, and hence $\omega_s + \omega_{-s} = \omega$. Since also the conjugacy class of s is $\{s, -s\}$, we conclude that

$$\chi([x, y]) = \sum_{s \in \mathcal{S}} \omega_s(x, y)c(s)\chi(s) = \frac{1}{2}\omega(x, y) \sum_{s \in \mathcal{S}} c(s)\chi(s) = 0,$$

which equals (3.2.2).

Next, fix $g \in \mathcal{S}$. Then, $V^g \neq 0$. Let $x \in V^g$ and $y \in V$ be such that $\omega(x, y) = 2$. Then,

$$\chi(g[x, y]) = \sum_{s \in \mathcal{S}} \omega_s(x, y)c(s)\chi(gs). \quad (3.3.1)$$

Let $S \subseteq \mathcal{S}$ be a conjugacy class of symplectic reflections. If $g \notin S$ then Proposition 2.0.1.(i) implies that $g \cdot S$ is again a conjugacy class in G . Hence

$$\begin{aligned}\omega_s(x, y)c(s)\chi(gs) + \omega_{-s}(x, y)c(-s)\chi(-gs) &= \omega(x, y)c(s)\chi(gs) \\ &= c(s)\chi(gs) + c(-s)\chi(-gs).\end{aligned}$$

If, on the other hand, $g \in S$, then $S = \{g, -g\}$, and our choice of x implies that $\omega_g(x, y) = 0$. Therefore $\omega_{-g}(x, y) = \omega(x, y)$ and hence

$$\omega_g(x, y)c(g)\chi(g^2) + \omega_{-g}(x, y)c(-g)\chi(-g^2) = 2c(-g)\chi(-\text{Id}).$$

Put together, (3.3.1) becomes

$$\chi(g[x, y]) = 2c(-g)\chi(-\text{Id}) + \sum_{s \in \mathcal{S} \setminus \{g, -g\}} c(s)\chi(gs),$$

implying (3.2.3).

4 The singular locus of $eH_c(G)e$

In this section, we will always take $V = \mathbf{C}^4$ and $G = Q_8 \times_{\mathbf{Z}/2} D_8$, except in Sect. 4.4, where we prove a result from the introduction.

It turns out to be possible to completely characterize the locus of $c \in \mathbf{C}$ such that $eH_c(G)e$ is singular, generalizing Theorem 1.2.2 (see Theorem 4.2.1 below). Before we do this, we recall some elementary facts about symplectic leaves, which are not strictly needed for the theorem, but which we will use to describe in more detail the singularities of those commutative spherical symplectic reflection algebras that are singular.

4.1 Recollections on symplectic leaves

Recall that an (algebraic) symplectic leaf of an affine Poisson variety X is a (Zariski) locally closed and connected smooth subvariety Y such that the tangent space $T_y Y$ at each point $y \in Y$ is spanned by Hamiltonian vector fields, $\xi_f := \{f, -\}$, for $f \in \mathbf{C}[X]$. The symplectic leaves are all symplectic manifolds (with Poisson structure obtained from the Poisson structure on X), and are in particular even-dimensional. When a Poisson variety X is a union of finitely many (necessarily disjoint) symplectic leaves, then this decomposition is unique. Moreover, the singular locus of X is exactly the union of those leaves that are not open in X (i.e., the positive-codimension leaves when X is irreducible). (We remark that this property of being a finite union of symplectic leaves is, in general, a strong condition, which was studied in, e.g., [10, 15]; note that it is always satisfied for varieties admitting a symplectic resolution.)

For every finite subgroup $G < \mathbf{Sp}(V)$, the Poisson variety $\text{Spec } \mathbf{C}[V^*]^G = V^*/G$ is a union of finitely many symplectic leaves, which are the G -orbits of the parabolic subspaces $(V^*)^K \subseteq V$ for subgroups $K < G$ (see, e.g., [4, Proposition 7.4]). Thus, for any filtered Poisson deformation A of $\mathbf{C}[V^*]^G$, it is also true that $\text{Spec } A$ has finitely many symplectic leaves: for each $i \geq 0$, the union of the $\leq 2i$ -dimensional leaves corresponds to a Poisson ideal $J \subseteq A$ whose associated graded Poisson ideal $\text{gr}(J)$ can only vanish on $\leq 2i$ -dimensional leaves of V^*/G . In particular, since $\text{gr}(J)$ is $\leq 2i$ -dimensional, so is J , and hence there can only be finitely many $2i$ -dimensional symplectic leaves of $\text{Spec } A$.

Therefore, in our situation where $V = \mathbb{C}^4$, describing the singularities of each commutative spherical symplectic reflection algebra deforming $\mathbb{C}[V^*]^G$ is equivalent to determining all two-dimensional and all zero-dimensional symplectic leaves.

Below, for our group $G = Q_8 \times_{\mathbb{Z}/2} D_8$, in addition to describing completely the set of parameters $c \in \mathbb{C}$ for which the corresponding algebra $eH_c(G)e$ is smooth (which by Theorem 1.2.2 forms an open subvariety of the parameter space), we will describe (and enumerate) all two-dimensional symplectic leaves of all commutative spherical symplectic reflection algebras (of which there are at most five, the maximum being obtained exactly for $\mathbb{C}[V^*]^G$ itself), and also give a bound (ten) on the number of zero-dimensional symplectic leaves of these algebras.

4.2 The parameters c for which $eH_c(G)e$ is singular

Recall that there are sixteen linear characters χ of the group G . These are uniquely specified by the constraints $\chi(s) = \pm 1$ for all $s \in \mathcal{S}$, $\chi(-\text{Id}) = 1$ and $\prod_{i=1}^5 \chi(s_i) = 1$ for a choice of representatives s_i of the conjugacy classes of \mathcal{S} .

Theorem 4.2.1 *The locus of $c \in \mathbb{C}$ such that $eH_c(G)e$ is singular is precisely the union of the following twenty-one hyperplanes:*

- (i) *The sixteen of the form $\sum_{s \in \mathcal{S}} \chi(s) \cdot c(s) = 0$, where χ is a one-dimensional character;*
- (ii) *The five of the form $c(s) = 0$ for some $s \in \mathcal{S}$ (equivalently, $c(-s) = 0$).*

In the case of type (i), $H_c(G)$ admits the one-dimensional representation χ with trivial action of V^ . In the case of type (ii), $H_c(G)$ admits two two-dimensional families of sixteen-dimensional irreducible representations whose G -structures are isomorphic to $\text{Ind}_{\{s,1\}}^G \mathbf{1}$ and $\text{Ind}_{\{s,1\}}^G \mathbf{sgn}$, respectively.*

Proof Choose $c \in \mathbb{C}$ such that $eH_c(G)e$ is not regular. Then $Z(H_c(G))$ is also not regular and we can choose a closed point $\psi : Z(H_c(G)) \rightarrow \mathbb{C}$ lying in the singular locus of $\text{Spec } Z(H_c(G))$. By Proposition 1.2.3, there exists an irreducible representation whose G -character χ is a proper subrepresentation of the regular representation. Then the parameter c satisfies Eqs. (3.2.2) and (3.2.3) and Lemma 4.2.3 implies that c must lie in one of the twenty-one hyperplanes in the statement of the theorem.

Conversely, if we choose c to lie in one of these twenty-one hyperplanes we must show that there exists a representation of $H_c(G)$ of dimension less than $|G|$. One can easily check the claim of part (i). Therefore we concentrate on part (ii) and assume that $c(s) = 0$ for some symplectic reflection s . Being a symplectic reflection, $\dim V^s = 2$. Let P be the parabolic subgroup of G that is the stabilizer of a generic point of V^s . Then $P = \langle s \rangle \simeq \mathbb{Z}_2$: if $p \in P$ then $V^s \subseteq V^p$ and either p is a symplectic reflection or $p = \text{Id}$. However, if p is a symplectic reflection not equal to s ($= s^{-1}$) then ps is neither Id nor a symplectic reflection. Now, consider the symplectic reflection algebra $H_{c|_P}(P, (V^P)^\perp)$ defined by P , the restriction $c|_P$ of c to P , and the symplectic vector space $(V^P)^\perp \subseteq V$. Since $c(s) = 0$, $H_{c|_P}(P, (V^P)^\perp) = H_0(\mathbb{Z}_2, \mathbb{C}^2)$. There exist (up to isomorphism) exactly two one-dimensional representations of $H_0(\mathbb{Z}_2, \mathbb{C}^2)$, which we denote by $L(\mathbf{1})$ and $L(\mathbf{sgn})$, which are isomorphic to the trivial and sign representations, respectively, as \mathbb{Z}_2 -modules, and have the trivial action of \mathbb{C}^2 . Part (ii) now follows from Losev's Theorems A.0.2 and A.0.3. In particular, the fact that there are two-dimensional families of representations of $H_c(G)$ isomorphic as G -modules to $\text{Ind}_{\{s,1\}}^G \mathbf{1}$ and $\text{Ind}_{\{s,1\}}^G \mathbf{sgn}$, respectively, follows from the fact that there is a two-dimensional leaf of $\text{Spec } Z(H_c(G))$, labeled by (P) such that, at each point of the

leaf, there are irreducible $H_c(G)$ -modules supported at that point isomorphic as G -modules to $\text{Ind}_{\{s,1\}}^G \mathbf{1}$ and $\text{Ind}_{\{s,1\}}^G \text{sgn}$. \square

Theorem 4.2.1 carries the following remarkable consequence, which is not otherwise obvious:

Corollary 4.2.2 *If $c(s) \equiv 1$ is the constant function, then $eH_c(G)e$ is smooth.*

Proof It is evident that $c(s) \equiv 1$ is not contained in any of the hyperplanes of type (ii) from Theorem 4.2.1. Also, since there are an odd number (5) of conjugacy classes of symplectic reflections $s \in S$, for every one-dimensional character χ of G , the number of occurrences of $+1$ among the values $\chi(s)$, $s \in S$ is not equal to the number of occurrences of -1 (and these are the only values that occur, since $s^2 = \text{Id}$ for all $s \in S$). Hence, the constant function $c(s) \equiv 1$ is not contained in any hyperplanes of type (i). Thus, the result follows from Theorem 4.2.1.

The following lemma, which is required in the proof of Theorem 4.2.1, is verified by computer.³

Lemma 4.2.3 *Let χ be the character of a proper G -submodule of the regular representation. Then the subspace of \mathbf{C} defined by equations (3.2.2) and (3.2.3) is contained in one of the twenty-one hyperplanes described in Theorem 4.2.1.*

4.3 The singular locus of singular $eH_c(G)e$

We can also deduce from the proof of Theorem 4.2.1 more information on the singularities of the singular $\text{Spec } eH_c(G)e$:

Corollary 4.3.1 *The number of two dimensional leaves in $\text{Spec } eH_c(G)e$ equals the number of conjugacy classes of symplectic reflections $\{s, -s\}$ such that $c(s) = 0$.*

Proof As noted in the proof of Theorem 4.2.1, the proper parabolic subgroups of G are all of the form $\langle s \rangle$ for some symplectic reflection s . Therefore there is a natural bijection $\{s, -s\} \mapsto (\langle s \rangle)$ between the conjugacy classes of symplectic reflections in G and conjugacy classes of proper parabolic subgroups of G . Let $P = \langle s \rangle$ and $\Xi = N_G(P)/P$. Now Losev's Theorem A.0.2 says that there is a bijection between height two Poisson prime ideals labeled by a conjugacy class (P) and the Ξ -orbits of maximal Poisson ideals in $Z_{c|_P}(\mathbf{Z}_2, \mathbf{C}^2)$. If $c(s) = 0$ then there is a unique maximal Poisson ideal in $Z_0(\mathbf{Z}_2, \mathbf{C}^2)$, which corresponds to the isolated singularity of $\mathbf{C}^2/\mathbf{Z}_2$. If $c(s) \neq 0$, then there are no maximal Poisson ideals in $Z_{c|_P}(\mathbf{Z}_2, \mathbf{C}^2)$. Therefore $c(s) = 0$ implies that there is a unique two-dimensional leaf in $\text{Spec } eH_c(G)e$ labeled by (P) and $c(s) \neq 0$ implies that there are no two-dimensional leaves labeled by (P) . \square

We can also give partial information on the zero-dimensional symplectic leaves of $\text{Spec } eH_c(G)e$. Recall that, for a Poisson algebra A , the zeroth Poisson homology is defined as $\text{HP}_0(A) := A/\{A, A\}$, where $\{A, A\}$ is considered as a vector subspace of A . The space of Poisson traces is the dual vector space, $\text{HP}_0(A)^* = \{\phi : A \rightarrow \mathbf{C} \mid \phi(\{a, b\}) = 0, \forall a, b \in A\}$. Recall also that, for each zero-dimensional symplectic leaf $\{x\} \subseteq \text{Spec } A$, evaluation at x is a Poisson trace, and these are linearly independent for distinct zero-dimensional leaves. Hence, the number of zero-dimensional symplectic leaves is at most $\dim \text{HP}_0(A)^*$.

³ The Magma [1] code used to verify Lemma 4.2.3 can be obtained by emailing the authors.

Proposition 4.3.2 *For all commutative spherical symplectic reflection algebras A deforming $\mathbb{C}[V^*]^G$ (with $G = Q_8 \times_{\mathbb{Z}/2} D_8$ and $V = \mathbb{C}^4$ as above), the following holds:*

- (a) $\dim \mathrm{HP}_0(A) = 10$;
- (b) $\mathrm{Spec} A$ has at most ten zero-dimensional symplectic leaves.

Note that part (a) confirms [9, Conjecture 1.3.5.(i)] on symplectic resolutions in this case, which states that, whenever V^*/G admits a symplectic resolution for $G < \mathrm{Sp}(V)$, then $\dim \mathrm{HP}_0(\mathbb{C}[V^*]^G)$ equals the number of conjugacy classes of elements $g \in G$ such that $g - \mathrm{Id}$ is invertible.

The result is perhaps surprising in that there is a very large number of proper subrepresentations of the regular representation of G ($5 \cdot 2^{16} - 1 = 327679$), and these can all be extended to representations of $H_c(G)$ at special values of c depending on the representation. Thus, in principle, at special values of c many of these could appear and be supported on many distinct zero-dimensional symplectic leaves. However, we see above that there are nonetheless at most ten zero-dimensional symplectic leaves at each value of c . (Note that, for example, at $c = 0$, all representations of G occur, but there is only one zero-dimensional symplectic leaf.)

Note also that this result does not rely on the brute-force computation underlying Lemma 4.2.3 (although it does rely on a different computer computation, namely computing $\mathrm{HP}_0(\mathbb{C}[V^*]^G)$ up to a certain polynomial degree provided by [8]).

Proof Using the methods of [8, §4] and Magma code there, we computed that $\mathrm{HP}_0(\mathbb{C}[V^*]^G)$ is ten-dimensional. Then, according to [8, Remark 2.13], ten is also an upper bound for $\dim \mathrm{HP}_0(A)$ for all $A = eH_c(G)e$. To show that $\dim \mathrm{HP}_0(A)$ is exactly ten for all such A , since the dimension is upper-semicontinuous, it suffices to show that it is ten-dimensional for generic c . Since $\mathrm{Spec} A$ is generically a symplectic manifold, this is a consequence of [7, Theorem 1.8.(ii)] together with the isomorphism $\mathrm{HP}_0(A) \cong H^{\dim \mathrm{Spec} A}(\mathrm{Spec} A)$ (as explained in [5, Theorem 22.2.1], more generally, $\mathrm{HP}_*(A) \cong H^{\dim \mathrm{Spec} A-*}(\mathrm{Spec} A)$ for symplectic $\mathrm{Spec} A$). \square

To summarize, if c does not lie on any of the twenty-one hyperplanes of Theorem 4.2.1 then $\mathrm{Spec} eH_c(G)e$ is a smooth symplectic manifold. If c is a generic point of one of the sixteen hyperplanes such that $c(s) \neq 0$ for all symplectic reflections s , then the singular locus of $\mathrm{Spec} eH_c(G)e$ consists of a single point, corresponding to a one-dimensional representation of $H_c(G)$ (with trivial action of V^*). If c lies on at least one of these sixteen hyperplanes but does not lie on any of the five hyperplanes $c(s) = c(-s) = 0$ for s a symplectic reflection, then the singular locus is zero-dimensional and consists of at most ten points. On the other hand, if $c(s) = 0$ for some s then in addition to the smooth locus, there are also two-dimensional and zero-dimensional leaves, with the number of two-dimensional leaves given by the number of hyperplanes of the form $c(s) = 0$ on which c lies (this is, obviously, at most five, with equality if and only if $eH_c(G)e = \mathbb{C}[V^*]^G$ itself), and again with at most ten zero-dimensional leaves. A generic point on one of the five hyperplanes of the form $c(s) = 0$ has exactly this corresponding two-dimensional leaf, and no other leaves aside from the open leaf.

We remark that we do not know how to compute precisely how many zero-dimensional symplectic leaves there are, nor even if the maximum of ten is attained for any c . To do this seems like an interesting problem (although it may be difficult, as it is analogous to determining the number, if any, of finite-dimensional representations admitted by a given quantization of $\mathbb{C}[V^*]^G$).

4.4 Proof of Proposition 1.2.3

The following observation was used in the proof of Proposition 1.2.3 to show that condition (ii) implies condition (i). This implication was required in order to make the computation reported in Lemma 4.2.3 tractable. In this subsection only, we allow V to be an arbitrary symplectic vector space and $G < \mathrm{Sp}(V)$ an arbitrary finite subgroup (G need no longer be the group $Q_8 \times_{\mathbf{Z}/2} D_8$).

Lemma 4.4.1 *Let $\chi : Z(H_c(G)) \rightarrow \mathbf{C}$ be a closed point of $\mathrm{Spec} Z(H_c(G))$. Then there exists a finite-dimensional $H_c(G)$ -module M such that M is isomorphic to the regular representation as a G -module and $z \cdot m = \chi(z)m$ for all $m \in M, z \in Z(H_c(G))$.*

Proof Let $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$ denote the variety of homomorphisms $\phi : H_c(G) \rightarrow \mathrm{End}_{\mathbf{C}}(\mathbf{C}G)$, whose restriction to G is the G -action of left multiplication. The group $\mathrm{Aut}_G(\mathbf{C}G)$ acts on $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$ by base change. It is shown in [7, Theorem 3.7] that there exists an irreducible component Rep° of $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$ such that the map $\pi : \mathrm{Rep}_{\mathrm{CG}}(H_c(G)) \rightarrow \mathrm{Spec} Z(H_c(G))$, sending a representation M to the algebra homomorphism $Z(H_c(G)) \rightarrow \mathbf{C}$ given by the action of $Z(H_c(G))$ on the line $eM \subseteq M$ (here $e = \frac{1}{|G|} \sum_{g \in G} g \in \mathbf{C}[G]$ is the symmetrizer element), restricts to an isomorphism of varieties $\mathrm{Rep}^\circ // \mathrm{Aut}_G(\mathbf{C}G) \xrightarrow{\sim} \mathrm{Spec} Z(H_c(G))$. The irreducible component Rep° is characterized as the closure in $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$ of the set $\mathrm{Rep}_{\mathrm{reg}}^\circ$ of points in $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$ that are irreducible $H_c(G)$ -modules. Fix a representative (M, ϕ_M) in the unique closed $\mathrm{Aut}_G(\mathbf{C}G)$ -orbit in $(\pi|_{\mathrm{Rep}^\circ})^{-1}(\chi)$ and write $M = M_1 \oplus \cdots \oplus M_k$ for the decomposition of M into irreducible $H_c(G)$ -modules. Without loss of generality, suppose $eM \subseteq M_1$. Then $z \cdot m = \chi(z)m$ for all $m \in M_1$. We need to show that $z \cdot m = \chi(z)m$ for all $m \in M$. Fix $z \in Z(H_c(G))$ and consider the closed subvariety

$$Y_z = \{\phi(z) - \pi(\phi)(z) \mathrm{Id}_{\mathbf{C}[G]} = 0 \mid \phi \in \mathrm{Rep}_{\mathrm{CG}}(H_c(G))\}$$

of $\mathrm{Rep}_{\mathrm{CG}}(H_c(G))$. Then $Y_z \cap \mathrm{Rep}^\circ$ is closed in Rep° . On the other hand, $\mathrm{Rep}_{\mathrm{reg}}^\circ \subseteq Y_z \cap \mathrm{Rep}^\circ$ which implies that $\mathrm{Rep}^\circ \subseteq Y_z$. Since $(M, \phi_M) \in \mathrm{Rep}^\circ$, this implies that $\phi_M(z) = \chi(z) \cdot \mathrm{Id}_M$, as desired. \square

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Appendix A: Summary of [18]

We summarize those results of [18] that have been used in this article (note that we only needed them for the proof of Theorem 4.2.1, not for the proof of Theorem 1.2.2 and hence also Theorem 1.1.2). For simplicity, we will use the notation $Z_c(G) := Z(H_c(G))$. Recall that a parabolic subgroup of G is defined to be any subgroup that is the stabilizer of some vector $v \in V$. Let P be a parabolic subgroup of G . By definition, it is normal in its normalizer $N_G(P)$. Let $\Xi := N_G(P)/P$ be the quotient. The algebra $H_c(G)$ has a canonical filtration given by placing G in degree zero and V in degree one. Then $Z_c(G)$ inherits this filtration by restriction and $\mathrm{gr}(Z_c(G)) \simeq \mathbf{C}[V^*]^G$. If $\mathfrak{p} \subset Z_c(G)$ is the prime ideal defining

the closure of a symplectic leaf of $\operatorname{Spec} Z_c(G)$, then it is known, by [20, Theorem 2.8], that $\operatorname{gr}(\mathfrak{p})$ is a prime ideal defining the closure of a symplectic leaf of V^*/G . Since the leaves of V^*/G are in bijection with conjugacy classes of parabolic subgroups of G (as recalled in Sect. 4.1), the leaves in $\operatorname{Spec} Z_c(G)$ can also be labeled by conjugacy classes of parabolic subgroups of G (however, the same conjugacy class could label several different leaves). Let $\operatorname{PSpec}_{(P)} Z_c(G)$ denote the set of all leaves in $\operatorname{Spec} Z_c(G)$ that are labeled by (P) , considered as a subset of $\operatorname{Spec} Z_c(G)$. We fix a representative P in each conjugacy class (P) . There is a unique zero-dimensional leaf $\{0\}$ in V^*/G ; it is labeled by (G) .

Next, we consider the algebra $H_{c|_P}(P, (V^P)^\perp)$, the symplectic reflection algebra defined by the subgroup P , the restriction $c|_P$, and the subspace $(V^P)^\perp \subseteq V$. The group Ξ acts on $\operatorname{Spec} Z_{c|_P}(P, (V^P)^\perp)$. Let $\operatorname{PSpec}_{(P)}^\Xi Z_{c|_P}(P, (V^P)^\perp)$ denote the set of Ξ -orbits of zero-dimensional leaves in $\operatorname{Spec} Z_{c|_P}(P, (V^P)^\perp)$. By [18, Theorem 1.3.2 (4)]:

Theorem A.0.2 *There exists a bijection $\operatorname{PSpec}_{(P)}^\Xi Z_{c|_P}(P, (V^P)^\perp) \xrightarrow{1:1} \operatorname{PSpec}_{(P)} Z_c(G)$.*

Now fix a leaf \mathcal{L} in $Z_c(G)$. It is labeled by some conjugacy class of parabolics, (P) say. For closed points $p \in \operatorname{Spec} Z_c(G)$ and $q \in \operatorname{Spec} Z_{c|_P}(P, (V^P)^\perp)$ corresponding to maximal ideals $\mathfrak{m}_p \subseteq Z_c(G)$ and $\mathfrak{n}_q \subseteq Z_{c|_P}(P, (V^P)^\perp)$, denote by $H_c(G)_p$ and $H_{c|_P}(P, (V^P)^\perp)_q$ the finite dimensional quotients of $H_c(G)$ and $H_{c|_P}(P, (V^P)^\perp)$ by the ideals generated by $\mathfrak{m}_p \subseteq Z_c(G)$ and \mathfrak{n}_q , respectively. Then [18, Theorem 1.4.1] says:

Theorem A.0.3 *Let $p \in \mathcal{L}$. Then there exists a zero-dimensional leaf $\{q\}$ in $\operatorname{Spec} Z_{c|_P}(P, (V^P)^\perp)$ and an isomorphism of finite-dimensional algebras*

$$\theta : H_c(G)_p \xrightarrow{\sim} \operatorname{Mat}_{|G/P|}(H_{c|_P}(P, (V^P)^\perp)_q)$$

such that the corresponding equivalence of categories $\theta^ : H_{c|_P}(P, (V^P)^\perp)_q - \operatorname{mod} \xrightarrow{\sim} H_c(G)_p - \operatorname{mod}$ satisfies*

$$\theta^*(M) = \operatorname{Ind}_P^G M$$

as G -modules.

Remark A.0.4 The second part of Theorem A.0.3 regarding G -module structures is not explicitly stated in [18, Theorem 1.4.1]. However, it follows from the definition of the isomorphism of [18, Theorem 2.5.3] and [2, Corollary 5.4].

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